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Wigner distributions on von Neumann lattices

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Abstract. Given an arbitrary Wigner distribution it is shown that by shifting it in a well-defined manner on a lattice in the phase plane one obtains a set of Wigner distributions which can be used for expanding any other Wigner distribution. For some particular class of functions, this set of Wigner distributions is orthogonal. The notion of such sets of distributions is extended to include frames on discrete lattices in phase plane.

1. Introduction

The canonical framework of classical mechanics with the coordinate x and momentum p being used together and on the same footing is very attractive, and already in the early stages of quantum mechanics there have been attempts to use this framework in quantum mechanics, despite the uncertainty principle. Thus, in the early 1930s Wigner introduced his distribution which is a function of two continuous variables in the xp-phase plane [1]. About the same time von Neumann had invoked the idea of defining a representation depending on two discrete variables which form a lattice in the xp-plane [2], known as the von Neumann representation.

There is a common feature shared by the Wigner distribution [1] and the von Neumann representation [2] in that both were first discovered in quantum mechanics and then independently they were rediscovered in signal processing by Ville [3] and by Gabor [4] in the $t\nu$-plane (t, time, ν, frequency), respectively. We focus here on the xp-plane but all the results can immediately be carried over into the $t\nu$-plane. In recent years, the Wigner distribution and the von Neumann representation have become useful tools in quantum mechanics [5, 6] and in signal processing [7, 8].

The Wigner distribution is defined directly as a function of the canonical pair x and p, and for any given wavefunction $\psi(x)$, it has the form

$$W_\psi(x, p) = \frac{1}{2\pi\hbar}\int \exp\left(-\frac{1}{\hbar}pz\right)\psi^*\left(x - \tfrac{1}{2}z\right)\psi\left(x + \tfrac{1}{2}z\right)\mathrm{d}z. \tag{1}$$

On the other hand, von Neumann defines a representation of $\psi(x)$ on a pair of discrete variables (m, n) in the phase plane xp by using the notion of a lattice

$$\alpha(m, n) = \frac{1}{\lambda\sqrt{2}}\left(am + \frac{1}{\hbar}\lambda^2 bn\right) \tag{2}$$

where m and n are integers, λ, a and b constants (λ and a have the dimension of x, and b the dimension of p), and ab is the area of the unit cell, which in the original von Neumann lattice was h, Planck's constant. The von Neumann representation (or Gabor's representation in signal processing) is defined by the expansion coefficients c_{mn} of $\psi(x)$ in a set of functions $g_{mn}(x)$ on the lattice of equation (2)

$$\psi(x) = \sum_{m,n} c_{mn} g_{mn}(x). \tag{3}$$

It was stated by von Neumann [2] and later proven in a number of works [9–11] that the functions $g_{mn}(x)$ for $ab = h$ form a complete set. The c_{mn} coefficients represent the wavefunction $\psi(x)$ on the discrete lattice and they form the von Neumann representation (or the Gabor representation in signal processing [12]). Since the Wigner distribution is quadratic in the wavefunction $\psi(x)$ (equation (1)) one should expect it to be related to products of the c_{mn} coefficients. These relations will, as a rule, contain cross-Wigner distributions which are defined on different wavefunctions on a lattice in the phase plane [12, 13]. As is well known [12, 14] for the von Neumann lattice $(ab = h)$ the series expansion in equation (3) is poorly convergent, and that the convergence is improved when $ab < h$. In the latter case the set g_{mn} in equation (3) turns into a frame [14, 15]. For finding the expansion coefficients c_{mn} both in the case of a von Neumann lattice $(ab = h)$ and in the case of $ab < h$, it is often convenient to use the kq-representation [12,14–17] which in signal processing is called the Zak transform [12].

In this paper we show how to construct a set of shifted Wigner distributions on a lattice in the phase plane which is complete in the sense that any other Wigner distribution can be expanded in this set in very much the same way as this was originally done for wavefunctions (see equation (3)). This construction is first carried out for the von Neumann case $(ab = h$ in equation (2)) and then extended to a set g_{mn} that forms a frame [14, 15]. In the case when $ab = h$, there is a special class of functions g_{mn} for which the Wigner distributions on a von Neumann lattice are orthogonal. Explicit formulae are established for the expansion coefficients both for the von Neumann case and for generalizations that form frames.

2. Shifted Wigner distributions

When working on lattices in phase plane it is convenient to use the shift operator [18]

$$D(x, p) = \exp\left(-\frac{\mathrm{i}}{2\hbar} xp\right) \exp\left(\frac{\mathrm{i}}{\hbar} p\hat{x}\right) \exp\left(-\frac{\mathrm{i}}{\hbar} x\hat{p}\right) \tag{4}$$

where $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators, and x and p are some values these operators can assume. By using the definition in equation (4) one can write the discrete set of functions in phase plane $g_{mn}(x)$ on a lattice (equation (3)) in the following way:

$$g_{mn}(x) = D(ma, nb)g(x) = \exp\left(-\frac{\mathrm{i}}{2\hbar} abmn\right) \exp\left(\frac{\mathrm{i}}{\hbar} xnb\right) g(x - ma) \tag{5}$$

where $g(x)$ is an arbitrary function. For $ab = h$ this is the original von Neumann set, while for $ab < h$ the set in equation (5) forms a frame [14].

For defining complete sets of Wigner distributions on lattices in phase plane we use the notion of a cross-Wigner distribution function [13] (also called a transition function [19]) which we define by means of the shift operator (equation (4))

$$W_{\psi_1,\psi_2}(x, p) = \frac{1}{\pi\hbar} \int (I\psi_1(z))^* D(-2x, -2p)\psi_2(z)\,\mathrm{d}z \tag{6}$$

where I is the inversion operator taking $\psi(z)$ into $\psi(-z)$. The definition in equation (6) is a generalization of the distribution $W_\psi(x, p)$ in equation (1) for a single function $\psi(x)$. An important fact about the cross-Wigner distribution is that when the functions ψ_1 and ψ_2 are obtained from the same function ψ by shifts in the phase plane by operators in equation (4), then the cross-distribution can be expressed by the Wigner distribution W_ψ for the single function ψ. After some simple calculations one finds

$$W_{D(x_1,p_1)\psi,D(x_2,p_2)\psi}(x,p) = \exp\left[\frac{\mathrm{i}}{\hbar}p(x_1-x_2) - \frac{\mathrm{i}}{\hbar}x(p_1-p_2) - \frac{\mathrm{i}}{2\hbar}(x_1p_2-x_2p_1)\right] \times W_\psi\left[x-\tfrac{1}{2}(x_1+x_2),\, p-\tfrac{1}{2}(p_1+p_2)\right]. \tag{7}$$

It is interesting that when $x_1 = x_2 = \bar{x}$ and $p_1 = p_2 = \bar{p}$ equation (7) simplifies and we obtain the known result [13]

$$W_{D(\bar{x},\bar{p})\psi,D(\bar{x},\bar{p})\psi}(x,p) = W_\psi(x-\bar{x},\, p-\bar{p}). \tag{8}$$

To us of particular interest is the case on lattices in the phase plane when $x_1 = m_1a$, $x_2 = m_2a$, $p_1 = n_1b$, $p_2 = n_2b$. Equation (7) then becomes

$$\begin{aligned} W_{D(m_1a,n_1b)\psi,D(m_2a,n_2b)\psi}(x,p) &= \exp\left[\frac{\mathrm{i}}{\hbar}(m_1-m_2)pa - \frac{\mathrm{i}}{\hbar}(n_1-n_2)xb - \frac{\mathrm{i}}{2\hbar}(m_1n_2-m_2n_1)ab\right] \\ &\quad\times W_\psi\left[x-\tfrac{1}{2}a(m_1+m_2),\, p-\tfrac{1}{2}b(n_1+n_2)\right]. \end{aligned} \tag{9}$$

We stress again the fact that the cross-Wigner distribution function for two functions at different points in the phase plane is expressed according to equations (7) and (9) by a Wigner distribution for a single function with x and p, respectively, shifted.

Having a function $\psi(x)$ which can be expanded in the set of functions $g_{mn}(x)$ of equation (5),

$$\psi(x) = \sum_{m,n} c_{mn} D(ma,nb)g(x) = \sum_{m,n} c_{mn}\exp\left(-\frac{\mathrm{i}}{2\hbar}abmn\right)\exp\left(\frac{\mathrm{i}}{\hbar}xnb\right)g(x-ma) \tag{10}$$

we ask the question about the expansion of the Wigner distribution $W_\psi(x,p)$ in a series of the elementary cross-Wigner distributions $W_{D(m_1a,n_1b)\psi,D(m_2a,n_2b)\psi}(x,p)$ on a lattice in phase plane. For this purpose we need to know another formula connecting the Wigner distribution for a linear combination $c_1\psi_1 + c_2\psi_2$, $W_{c_1\psi_1+c_2\psi_2}(x,p)$ with the cross-Wigner distributions

$$W_{c_1\psi_1+c_2\psi_2}(x,p) = \sum_{i,j=1}^{2} c_i^* c_j W_{\psi_i,\psi_j}. \tag{11}$$

With the formulae in equations (9)–(11) at hand one easily finds the following expansion:

$$\begin{aligned} W_\psi(x,p) &= \sum_{\substack{mn\\ m'n'}} c_{mn}^* c_{m'n'} W_{g_{mn},g_{m'n'}}(x,p) \\ &= \sum_{\substack{mn\\ m'n'}} c_{mn}^* c_{m'n'} \exp\left[\frac{\mathrm{i}}{\hbar}(m-m')pa - \frac{\mathrm{i}}{\hbar}(n-n')xb - \frac{\mathrm{i}}{2\hbar}(mn'-m'n)ab\right] \\ &\quad\times W_g\left[x-\tfrac{1}{2}a(m+m'),\, p-\tfrac{1}{2}b(n+n')\right]. \end{aligned} \tag{12}$$

This formula shows that given the Wigner distribution $W_g(x,p)$ for the function $g(x)$ one can expand any other Wigner distribution $W_\psi(x,p)$ in the set of shifted distributions

$$W_g\left(x-\tfrac{1}{2}ak,\, p-\tfrac{1}{2}b\ell\right) \tag{13}$$

with k and ℓ assuming all integer values. Clearly, for this one has to know the expansion coefficients which, in principle, can be determined from the expansion in equation (10), but as is shown below they can also be found directly from the Wigner distributions W_ψ and W_g. The expansions in equations (10) and (12) for the wavefunction $\psi(x)$ and for the Wigner distribution $W_\psi(x, p)$ are of the same nature. In the case of $\psi(x)$ one starts with the function $g(x)$ (called the test function) and by shifting it on a discrete lattice in the phase plane one arrives at to a set of functions that can be used for expanding $\psi(x)$ (equation (10)). The result of equation (12) is similar: one starts with the Wigner distribution $W_g(x, p)$ (which can be called the test Wigner distribution) and one builds a set of shifted Wigner distributions in the phase plane (equation (13)) which can be used for expanding any arbitrary Wigner distribution $W_\psi(x, p)$ according to equation (12). It is interesting to point out, however, that when the area of the unit cell for the lattice in the phase plane in the expansion of $\psi(x)$ (equation (10)) is ab, the unit cell area for the expansion of $W_\psi(x, p)$ is $\frac{1}{4}ab$ (equations (12) and (13)). The appearance of a $\frac{1}{4}$ unit cell seems to be characteristic for Wigner distributions on lattices in the phase plane [12, 20].

3. Expansion coefficients

For finding the expansion coefficients of the expansions in equation (10) or (12) (the latter are products of the former), we consider first in detail the von Neumann case, $ab = h$ (see equation (2)), and the results will then be generalized to frames [14, 15] with $ab = h/N$ where N is an integer $\geqslant 2$. When working on lattices it is convenient to use the kq transform [16]

$$
\begin{aligned}
C_\psi^{(d)}(k, q) &= \left(\frac{d}{2\pi}\right)^{1/2} \sum_n \exp(\mathrm{i}kdn)\psi(q - nd) \\
\psi(x) &= \left(\frac{d}{2\pi}\right)^{1/2} \int_{-\pi/d}^{\pi/d} C_\psi^{(d)}(k, x)\,\mathrm{d}k
\end{aligned}
\tag{14}
$$

where d is an arbitrary constant. In the von Neumann case $b = h/a$ (see equation (12)), and the set in equation (5) then becomes in the kq-representation (we put $d = a$)

$$
C_{mn}^{(a)}(k, q) = (-1)^{mn} \exp\left(-\mathrm{i}kam + \mathrm{i}\frac{2\pi}{a}qn\right) C_g^{(a)}(k, q). \tag{15}
$$

It is also useful to define the tilde set [12, 16, 21]:

$$
\tilde{C}_{mn}^{(a)}(k, q) = (-1)^{mn} \exp\left(-\mathrm{i}kam + \mathrm{i}\frac{2\pi}{a}qn\right) C_{\tilde{g}}^{(a)}(k, q) \tag{16}
$$

with

$$
C_{\tilde{g}}(k, q) = \frac{1}{2\pi C_g^*(k, q)}. \tag{17}
$$

The tilde set can in some sense be called biorthogonal to the original von Neumann set $C_{mn}(k, q)$ in that they satisfy the following biorthogonality relation:

$$
\int \tilde{C}_{mn}^*(k, q) C_{m'n'}(kq)\,\mathrm{d}k\,\mathrm{d}q = \delta_{mm'}\delta_{nn'}. \tag{18}
$$

One should, however, have in mind that $C_{\tilde{g}}(k, q)$ is, in general, not square integrable [12, 21]. In the x-representation the tilde function $\tilde{g}(x)$ is (see equation (14))

$$
\tilde{g}(x) = \left(\frac{a}{2\pi}\right)^{1/2} \int C_{\tilde{g}}(k, x)\,\mathrm{d}k. \tag{19}
$$

As is well known, the expansion coefficients c_{mn} in equation (10) for the von Neumann case $(ab = h)$ can be expressed in the kq-representation in the following way [12, 17, 21]:

$$c_{mn} = \frac{1}{2\pi} \iint \tilde{C}^{(a)}_{mn}(k, q) C(k, q) \, \mathrm{d}k \, \mathrm{d}q \tag{20}$$

where $\tilde{C}^{(a)}_{mn}(k, q)$ is given in equation (16) and the integration is over a unit cell in the von Neumann lattice: from 0 to $2\pi/a$ for k and from 0 to a for q. In the expansion of the Wigner distribution in equation (12) the expansion coefficients are products $c^*_{mn} c_{m'n'}$ and in what follows we show how to express them directly by integrals on Wigner functions. For this we use Moyal's formula [13, 21]

$$\int W^*_{\psi_1,\psi_2}(x, p) W_{\psi_3,\psi_4}(x, p) \, \mathrm{d}x \, \mathrm{d}p = \frac{1}{2\pi\hbar} (\psi_3, \psi_1)(\psi_2, \psi_4). \tag{21}$$

From here the very handy orthogonality relation follows for the Wigner distribution functions (we use the orthogonality of $\tilde{g}_{mn}$ and $g_{m'n'}$ in equation (18)):

$$\int W^*_{D(ma,nb)\tilde{g},D(m'a,n'b)\tilde{g}}(x, p) W_{D(ka,\ell b)g,D(k'a,\ell' b)g}(x, p) \, \mathrm{d}x \, \mathrm{d}p = \delta_{mk}\delta_{n\ell}\delta_{m'k'}\delta_{n'\ell'}. \tag{22}$$

The expansion coefficients in equation (12) for $ab = h$ are then obtained by multiplying the latter on both sides by $W^*_{D(ka,\ell b)\tilde{g},D(k'a,\ell' b)\tilde{g}}(x, p)$ and by integrating over x and p. We have

$$c^*_{k\ell} c_{k'\ell'} = 2\pi\hbar \iint W^*_{D(ka,\ell b)\tilde{g},D(k'a,\ell' b)\tilde{g}}(xp) W(x, p) \, \mathrm{d}x \, \mathrm{d}p \tag{23}$$

where we used the orthogonality relation in equation (22).

In general, the functions in the von Neumann set in equation (5) with $ab = h$ are non-orthogonal, and they satisfy a biorthogonality relation in the form of equation (18). However, there is a very wide class of square-integrable functions $C(k, q)$ for which the von Neumann set is orthogonal. These are functions whose $C(k, q)$'s satisfy the condition [11, 22, 23]

$$|C(k, q)| = \text{constant}. \tag{24}$$

For this class of functions when $C(k, q)$ is chosen to be of norm 1, the tilde function equals the kq-function itself (equation (17))

$$\tilde{C}(k, q) = C(k, q) \tag{25}$$

and it then follows that the orthogonality relations in equations (18) and (22) hold for the functions themselves (the tildes can be erased). Correspondingly, the expansion coefficients in equations (20) and (23) will also have no tilde functions.

Finally, let us remark on the extension of the above results to frames. As was mentioned above the case of the von Neumann lattice with $ab = h$ leads to poorly convergent series [12, 14, 21]. The convergence is improved considerably for sets $g_{mn}(x)$ (equation (5)) that form frames, and in what follows we discuss the case when [14]

$$ab = \frac{h}{N} \qquad N = 2, 3, \ldots. \tag{26}$$

For this purpose we choose an arbitrary constant d and two integers L and M with their product $LM = N$. We then write

$$a = \frac{d}{L} \qquad \text{and} \qquad b = \hbar \frac{2\pi}{Md} \tag{27}$$

which ensures that $ab = h/N$ as in equation (26). By using this notation we define the frame operator [15] in the kq-representation

$$F(k,q) = 2\pi \sum_{s=1}^{M}\sum_{t=1}^{L}\left|g^{(d)}\left(k - \frac{2\pi}{Md}s, q - \frac{d}{L}t\right)\right|^2. \tag{28}$$

This operator is periodic under all the translations $D(ma, nb)$ in equation (5), and the von Neumann set (equation (15)) can be generalized to

$$\phi_{mn}^{(d)}(k,q) = \frac{g_{mn}^{(d)}(k,q)}{F^{1/2}(k,q)} \tag{29}$$

where $g_{mn}^{(d)}(k,q)$ is the kq transform of the set of functions in equation (5), and the relations between a, b and d are given in equation (27). The advantage of using the new functions $\phi_{mn}^{(d)}(k,q)$ is that for them a convenient decomposition of the unit operator I exists [14, 15],

$$\sum_{m,n} |\phi_{mn}^{(d)}\rangle\langle\phi_{mn}^{(d)}| = I. \tag{30}$$

An immediate consequence of this decomposition is the expansion of any function $C(k,q)$ in the set $\phi_{mn}(k,q)$

$$C(k,q) = \sum_{m,n} c_{mn}^{(f)}\phi_{mn}^{(d)}(k,q) \tag{31}$$

where the expansion coefficients are (f stands for frame)

$$c_{mn}^{(f)} = \int \phi_{mn}^{*(d)}(k,q)C(k,q)\,\mathrm{d}k\,\mathrm{d}q. \tag{32}$$

Correspondingly, the expansion of any Wigner distribution will hold as given by equation (12) but with c_{mn} replaced by $c_{mn}^{(f)}$ and with g replaced by ϕ (equation (29)). The direct calculation of the expansion coefficients in equation (23) will also hold but with $\tilde{g}$ replaced by ϕ, as is easily seen by using equation (32) and Moyal's formula (21).

4. Example and conclusions

As an elementary example of constructing a complete set of Wigner distributions (equation (13)), let us consider the ground state $\psi_0(x)$ of the harmonic oscillator

$$\psi_0(x) = \left(\frac{1}{\lambda\sqrt{\pi}}\right)^{1/2}\exp\left(-\frac{x^2}{2\lambda^2}\right) \tag{33}$$

where λ is a constant. By using the definition in equation (1), the Wigner function $W_0(x, p)$ for this ground state is

$$W_0(x,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{x^2}{\lambda^2} - \frac{\lambda^2 p^2}{\hbar^2}\right). \tag{34}$$

We restrict ourselves to the case of a von Neumann lattice ($ab = h$), and then the complete set of Wigner distributions according to equation (13) becomes

$$W_0\left(x - \frac{a}{2}k,\ p - \frac{\pi}{a}\hbar\ell\right) = \frac{1}{\pi\hbar}\exp\left[-\frac{(x-(q/2)k)^2}{\lambda^2} - \frac{\lambda^2(p-(\pi/a)\hbar\ell)^2}{\hbar^2}\right] \tag{35}$$

where k and ℓ equal $0, \pm 1, \pm 2, \ldots$. From what we have proven above the system of Wigner functions in equation (35) is complete in the sense that any Wigner function $W_\psi(x, p)$ can be expanded in it according to equation (12) (we use the condition for a von Neumann lattice $ab = h$):

$$W_\psi(x, p) = \sum_{\substack{m,n \\ m'n'}} (-1)^{mn'-m'n} c^*_{mn} c_{m'n'} \exp\left[\frac{\mathrm{i}}{\hbar}(m - m')pa - \mathrm{i}(n - n')x\frac{2\pi}{a}\right] \times W_0\left[x - \frac{a}{2}(m + m'),\ p - \frac{\pi\hbar}{a}(n + n')\right] \tag{36}$$

where W_0 is given in equation (35). To find the expansion coefficients c_{mn} we can use the expression in equation (20), and for this we need the tilde set of equation (16) for the ground state of a harmonic oscillator. The kq-function $C_0(k, q)$ for $\psi_0(x)$ in equation (33) is according to equation (14)

$$C_0^{(a)}(k, q) = \left(\frac{a}{2\lambda\pi^{3/2}}\right)^{1/2} \exp\left(-\frac{q^2}{2\lambda^2}\right)\theta_3(z|\sigma) \tag{37}$$

where θ_3 is the well known theta function

$$\theta_3(z|\sigma) = \sum_{n=-\infty}^{\infty} \exp(2\mathrm{i}zn + \mathrm{i}\pi\sigma^2 n^2) \tag{38}$$

and

$$z = \frac{ka}{2} - \mathrm{i}\frac{qa}{2\lambda^2} \qquad \sigma = \mathrm{i}\frac{a^2}{2\pi\lambda^2}. \tag{39}$$

With this in mind we obtain the following expression for the tilde set of equation (16) for the ground state of a harmonic oscillator:

$$C_{mn}^{(a)}(k, q) = (-1)^{mn} \exp\left(-\mathrm{i}kam + \mathrm{i}\frac{2\pi}{a}qn\right)\tilde{C}_0^{(a)}(k, q) \tag{40}$$

where according to equation (17) $\tilde{C}_0^{(a)}(k, q)$ is

$$\tilde{C}_0^{(a)}(k, q) = \frac{1}{2\pi C_0^*(k, q)}. \tag{41}$$

We now have all the information needed for calculating the expansion coefficients c_{mn} for the example in equation (36). For this we use the formula given by equation (20) with the kq-function $C(k, q)$ for the wavefunction $\psi(x)$ which is obtained according to equation (14).

In conclusion, the main result of this paper is that any Wigner distribution $W_\psi(x, p)$ for an arbitrary state $\psi(x)$ can be expanded in a set of elementary Wigner distributions that are obtained by starting with a fixed Wigner distribution $W_g(x, p)$ for an appropriately chosen function $g(x)$, and by shifting it on a lattice in the phase plane according to equation (12). In view of the fact that $W_g(x, p)$ is in our disposition, it is to be expected that the expansion (12) will find applications both in physics and in signal processing.

We have considered in detail the Wigner distribution but there are many other distributions of position x and momentum p available which were studied in detail in the literature [24] and in a recent basic book [25]. A convenient list of relations between the different distributions is summarized in a recent publication [26]. Our expansion technique on von Neumann lattices can without much difficulty be extended to the variety of other distributions in the phase plane. An important physical situation which brings about a von Neumann lattice in a natural way is the problem of an electron in a constant magnetic field. The relevant phase plane in this problem is related to the plane of orbital centres for the motion perpendicular to the magnetic field. In recent publications [23, 27] the von Neumann lattice was used to construct a complete set of orthonormal states for each Landau level. One should expect that the expansion scheme of Wigner distributions (or any other phase plane distributions) on von Neumann lattices will be well applicable to the magnetic field problem when combined with the complete set of orthonormal states for Landau Levels. It should be pointed out that in signal processing there are other works that have previously considered the relations between the von Neumann (in signal processing it is Gabor) and Wigner representations [21, 28].

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